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ON THE STABLE DERIVATION ALGEBRA ASSOCIATED WITH SOME BRAID GROUPS

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ABSTRACT

We shall prove some stability property of the graded Lie algebra $\mathcal{D}_n$ of certain derivations associated with pure sphere braid group on n strings; in fact, that $\mathcal{D}_n \simeq \mathcal{D}_5$ for $n \geq 6$. These Lie algebras $\mathcal{D}_n$ are connected with some big l-adic Galois representations, and the stability property is related to some conjecture of Grothendieck.

Introduction

Let $\mathfrak{P}_n (n \geq 4)$ be the graded Lie algebra over $\mathbf{Q}$ associated with the lower central series of the pure sphere braid group on n strings, and $\mathcal{D}_n$ be the graded Lie algebra over $\mathbf{Q}$ consisting of all "S_n-invariant special" outer derivations of $\mathfrak{P}_n$ (see §1 below). This algebra $\mathcal{D}_n$ has drawn our attention in connection with the action of the Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on the pro-l fundamental group of $\mathbf{P}^1 - \{0, 1, \infty\}$. A certain basic Galois Lie algebra $\mathfrak{g}^{(l)}$ associated with this action is *contained in* $\mathcal{D}_n \otimes \mathbf{Q}_l$ for each $n \geq 4$ and each prime l ([5]§5). The structure of $\mathfrak{P}_n$ was determined by T. Kohno [8] (see §1.1 below), but as for $\mathcal{D}_n$, we know much less. There are natural sequences of projections

$$\to \mathfrak{P}_n \to \cdots \to \mathfrak{P}_5 \to \mathfrak{P}_4,$$
$$\to \mathcal{D}_n \to \cdots \to \mathcal{D}_5 \to \mathcal{D}_4,$$

in which the arrows $\mathfrak{P}_n \to \mathfrak{P}_{n-1}$ are surjective (with big kernels), while $\mathcal{D}_n \to \mathcal{D}_{n-1}$ are **injective** [4] (cf. [7] for some generalizations), both for $n \geq 5$. The main

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purpose of this paper is to prove that $\mathcal{D}_n \to \mathcal{D}_{n-1}$ is **bijective** for $n \geq 6$. (This gives an affirmative answer to the question "$\mathcal{D}_5 = \mathcal{D}_\infty$?" raised in [5] (Q5.3.4(i)).) Thus,

$$\cdots \xrightarrow{\sim} \mathcal{D}_n \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{D}_5 \hookrightarrow \mathcal{D}_4$$

(and $\mathcal{D}_5 \not\twoheadrightarrow \mathcal{D}_4$; see §1.2). It is an open question whether $\mathfrak{g}^{(l)} \simeq \mathcal{D}_5 \otimes \mathbf{Q}_l$ and (hence) $\mathcal{D}_5$ gives a common $\mathbf{Q}$-structure for the l-adic Lie algebras $\mathfrak{g}^{(l)}$. This stability property may be regarded as a graded Lie algebra version, in the case of genus 0, of a more general property of the "Teichmüller Lego" *predicted* by Grothendieck [2] (see also [5] §3.3, §5.3; [1] §4, especially a question raised a few lines after the formula (4.13)).

The main results are: Main Theorem (§1.2), Theorem 1 (§2.4), Theorem 2 and Proposition 9 (§4.2).

About the proofs. Since the injectivity of $\mathcal{D}_n \to \mathcal{D}_{n-1} (n \geq 5)$ was already established [4], the question is the extendability of each element of $\mathcal{D}_5$ to that of $\mathcal{D}_n$ $(n \geq 6)$. The author obtained the first proof of the extendability by using the action of the Grothendieck–Teichmüller group $\mathrm{GT}(k)$ on $B_n(k)$ defined in Drinfeld [1]. Here, k is some field of characteristic $0, B_n$ is the plane braid group on n strings, and $B_n(k)$ is a certain "k-nilpotent completion" associated with B_n. The graded Lie algebra of $\mathrm{GT}_1(k) (\subset \mathrm{GT}(k))$ is isomorphic to $\mathcal{D}_5 \otimes k$ (compare our Theorem 1 with [1] §§5, 6), and it can be checked that the above action induces an "n-compatible" system of Lie algebra homomorphisms $\mathcal{D}_5 \to \mathcal{D}_n$ $(n \geq 5)$. This leads directly to the extendability. But in this proof, verifications of some technical points are fairly involved and lengthy. We shall therefore choose another way and give a proof which lies within the framework of graded Lie algebras.

1. Definitions and the statement of the main result

1.1 The graded Lie algebra $\mathfrak{P}_n$ over $\mathbf{Q}$ $(n \geq 4)$ has the following presentation:

Generators x_{ij} $(1 \leq i, j \leq n)$;

Relations (i) $x_{ii} = 0$ $(1 \leq i \leq n)$, $x_{ij} = x_{ji}$ $(1 \leq i, j \leq n)$;

(ii) $\sum_{j=1}^{n} x_{ij} = 0$ $(1 \leq i \leq n)$;

(iii) $[x_{ij}, x_{kl}] = 0$ if $\{i, j\} \cap \{k, l\} = \phi$.

The grading $\deg(x_{ij}) = 1$ $(1 \leq i, j \leq n)$.

We denote by $\mathrm{gr}^m\mathfrak{P}_n$ the homogeneous part of $\mathfrak{P}_n$ of degree m ($m \geq 1$). It is easy to see that $x_{ij} + x_{jk} + x_{ki}$ commutes with x_{ij}, x_{jk}, x_{ki} for any indices i, j, k, and that

$$x_{ij} = \sum{}' x_{kl}, \tag{1.1.1}$$

where the summation $\sum'$ is over all indices k, l with $k < l$ and $\{k,l\} \cap \{i,j\} = \phi$.

The symmetric group S_n acts on $\mathfrak{P}_n$ via $x_{ij} \to x_{\sigma i, \sigma j}$ ($\sigma \in S_n$), inducing a linear action on $\mathrm{gr}^m\mathfrak{P}_n$ for each m.

When $n = 4$, one has, by (1.1.1), $x_{12} = x_{34}(:= x)$, $x_{23} = x_{14}(:= y)$, $x_{13} = x_{24}(:= z)$, with $x + y + z = 0$, and $\mathfrak{P}_4$ is a free Lie algebra on x, y. The group S_4 acts on $\mathfrak{P}_4$ through its quotient $\simeq S_3$ as substitutions of x, y, z.

When $n \geq 5$, $\mathfrak{P}_n$ is a successive extension of free graded Lie algebras of ranks $2, 3, \ldots, n-2$. To see this, let N_i ($1 \leq i \leq n$) denote the Lie subalgebra of $\mathfrak{P}_n$ generated by $x_{i1}, \ldots, x_{in}$. Then ([4]; Prop 3.2.1, its proof and Prop 3.3.1) N_i is an *ideal*, which is free of rank $n-2$, being generated by any $n-2$ members among the x_{ij} ($1 \leq j \leq n, j \neq i$). Moreover, $\mathfrak{P}_n/N_i \simeq \mathfrak{P}_{n-1}$. Therefore, $\mathfrak{P}_n$ is a successive extension of free graded Lie algebras (of ranks $2, 3, \ldots, n-2$). In particular, it has trivial center. For each i, j ($1 \leq i, j \leq n$), $i \neq j$, let C_{ij} denote the centralizer of x_{ij} in $\mathfrak{P}_n$. Then ($loc \cdot cit$) C_{ij} is generated by x_{kl} for $\{k,l\} \cap \{i,j\} = \phi$, and

$$N_i + C_{ij} = \mathfrak{P}_n, \quad N_i \cap C_{ij} = \mathbf{Q}x_{ij} \ (\subset \mathrm{gr}^1\mathfrak{P}_n). \tag{1.1.2}$$

In particular,

$$\mathrm{gr}^m\mathfrak{P}_n = \mathrm{gr}^m N_i \oplus \mathrm{gr}^m C_{ij} \quad (m > 1). \tag{1.1.3}$$

This decomposition will be often used later.

1.2 A derivation of $\mathfrak{P}_n$ is a $\mathbf{Q}$-linear endomorphism D of $\mathfrak{P}_n$ such that

$$D([y, y']) = [Dy, y'] + [y, Dy'] \quad (y, y' \in \mathfrak{P}_n).$$

It is called **special** if for each i, j ($1 \leq i, j \leq n$) there exists some $t_{ij} \in \mathfrak{P}_n$ such that $D(x_{ij}) = [t_{ij}, x_{ij}]$. Special derivations of $\mathfrak{P}_n$ form a graded Lie algebra; the degree m part consists of those D with $t_{ij} \in \mathrm{gr}^m\mathfrak{P}_n$ (all i, j), and $[D, D'] := D \circ D' - D' \circ D$. This algebra contains the inner derivations as homogeneous ideal, and the quotient will be called the (graded Lie) algebra of **special outer** derivations. If D is a derivation of $\mathfrak{P}_n$ and $\sigma \in S_n$, then $\sigma \circ D \circ \sigma^{-1}$ is again a derivation. This $D \to \sigma \circ D \circ \sigma^{-1}$ induces an S_n-action on the algebra of special outer derivations. We define $\mathcal{D}_n$ to be the graded Lie algebra over $\mathbf{Q}$ consisting of all S_n-invariant special outer derivations of $\mathfrak{P}_n$.

Now let $n \geq 5$. Then each special derivation D of $\mathfrak{P}_n$ leaves the kernel $N_n = \langle x_{n1}, \cdots, x_{n,n-1}\rangle$ of the projection $\mathfrak{P}_n \to \mathfrak{P}_{n-1}$ defined by $x_{ij} \to x_{ij}$ $(1 \leq i, j \leq n-1)$ stable, and hence D induces a special derivation $\overline{D}$ of $\mathfrak{P}_{n-1}$. This $D \to \overline{D}$ induces a homomorphism $\psi_n : \mathcal{D}_n \to \mathcal{D}_{n-1}$. We have shown [4] that ψ_n is *injective* ($n \geq 5$). The main goal of this note is to give a proof of:

MAIN THEOREM: *ψ_n is bijective for $n \geq 6$.*

Thus, ψ_n induces:

$$\stackrel{\sim}{\to} \mathcal{D}_n \stackrel{\sim}{\to} \cdots \stackrel{\sim}{\to} \mathcal{D}_5 \hookrightarrow \mathcal{D}_4.$$

Ihara–Terada and Drinfeld have independently verified that $\dim \mathrm{gr}^7 \mathcal{D}_5 = 1 < 2 = \dim \mathrm{gr}^7 \mathcal{D}_4$ ([1][6]).

Remark 1: It is easy to see that $\mathrm{gr}^1 \mathcal{D}_4 = (0)$. Therefore, $\mathrm{gr}^1 \mathcal{D}_n = (0)$ for all $n \geq 4$, by the injectivity of ψ_n. So, in the following study of $\mathrm{gr}^m \mathcal{D}_n$, we can restrict ourselves to the case $m > 1$.

2. The y-normalization; $\mathcal{D}_4$ and $\mathcal{D}_5$

2.1 For each $n \geq 4$, we shall make use of the following set of elements $y_i = y_i^{(n)}$ of $\mathfrak{P}_n$;

$$y_i = \sum_{j=1}^{i-1} x_{ij} = -\sum_{j=i+1}^{n} x_{ij} \quad (2 \leq i \leq n-1). \tag{2.1.1}$$

Then, clearly, $y_2, \ldots, y_{n-1}$ are mutually commutative, $y_1 = x_{12}$, $y_{n-1} = -x_{n-1,n}$, and

$$y_2 + \cdots + y_{n-1} = 0. \tag{2.1.2}$$

PROPOSITION 1:

(i) *If $z \in \mathrm{gr}^m \mathfrak{P}_n$ $(m > 1)$ commutes with all y_i $(2 \leq i \leq n-1)$, then $z = 0$;*

(ii) *Each class of special derivations modulo inner derivations of $\mathfrak{P}_n$ of degree $m > 1$ contains a unique derivation D such that*

$$D(y_2) = \cdots = D(y_{n-1}) = 0. \tag{2.1.3}$$

Proof:

(i) Induction on $n \geq 4$. For $n = 4$, $\mathfrak{P}_4$ is free on $x(= x_{12})$, $y(= x_{23})$, and $y_2 = -y_3 = x$. But the centralizer of x in $\mathfrak{P}_4$ is $\mathbf{Q}x \subset \mathrm{gr}^1\mathfrak{P}_4$; hence this is valid for $n = 4$. Now let $n \geq 5$ and assume that (i) is valid for $n-1$. Let $z \in \mathrm{gr}^m\mathfrak{P}_n$ $(m > 1)$ commute with $y_i = y_i^{(n)}$ $(2 \leq i \leq n-1)$. Then, since the projection of $y_i^{(n)}$ on $\mathfrak{P}_{n-1} = \mathfrak{P}_n/N_n$ is $y_i^{(n-1)}$ $(2 \leq i \leq n-2)$, the induction assumption implies that the projection of z on $\mathfrak{P}_{n-1}$ must vanish; hence $z \in N_n$. But since z commutes also with $y_{n-1}^{(n)} = -x_{n-1,n}$, and N_n is free on $x_{2n}, \ldots, x_{n-1,n}$ (and moreover $\deg z > 1$), z must be 0.

(ii) The uniqueness is obvious by (i). As for the existence, we shall not need the assumption $m > 1$. We proceed by induction. When $n = 4$, (ii) is obvious, as $y_2 = -y_3 = x_{12}$.

Now let $n \geq 5$ and assume that (ii) is valid for $n-1$. Then, by using the projection $\mathfrak{P}_n \to \mathfrak{P}_{n-1}$ and the induction assumption, we see easily that a given class (modulo inner derivations) contains such a derivation D' that $D'(y_i) \in N_n$ $(2 \leq i \leq n-2)$. As D' is special and $y_{n-1} = -x_{n-1,n}$, $D'(y_{n-1}) = [t', y_{n-1}]$ with some $t' \in \mathfrak{P}_n$. As $\mathfrak{P}_n = C_{n-1,n} + N_n$, we may assume $t' \in N_n$. Put $D = D' - \mathrm{Int}(t')$ ($\mathrm{Int}(t')$: the inner derivation $* \mapsto [t', *]$). Then $D(y_{n-1}) = 0$, and as $t' \in N_n$ and N_n is an ideal, $D(y_i) \in N_n$ $(2 \leq i \leq n-2)$. But since $[y_i, y_{n-1}] = 0$ and $D(y_{n-1}) = 0$, we have $[D(y_i), y_{n-1}] = 0$. Therefore, $D(y_i) \in C_{n-1,n} \cap N_n$. As $\deg D(y_i) > \deg y_i = 1$, we have $D(y_i) = 0$ also for $2 \leq i \leq n-2$. Therefore, D satisfies the required property. ∎

A special derivation D of $\mathfrak{P}_n$ will be called y-**normalized** if it satisfies (2.1.3). By Proposition 1 (and Remark 1), each element of $\mathcal{D}_n$ is represented by a unique y-normalized special derivation D. The corresponding element of $\mathcal{D}_n$ will be denoted by $\{D\}$. Note that if D, D' are y-normalized, then so is $[D, D']$. Thus, $\mathcal{D}_n$ is isomorphic to the algebra of all those y-normalized special derivations of $\mathfrak{P}_n$ that are S_n-invariant modulo inner derivations.

As a representative modulo inner derivations for each element of $\mathcal{D}_n$, we may also choose an S_n-invariant derivation (which is unique only up to inner derivations w.r.t. S_n-invariant elements of $\mathfrak{P}_n$). But it seems that the y-normalized representative is more useful for our present purpose.

2.2 The case $n = 4$. Recall that $\mathfrak{P}_4$ is free on $x = x_{12} = x_{34}$ and $y = x_{23} = x_{14}$, and $x + y + z = 0$ for $z = x_{13} = x_{24}$.

PROPOSITION 2:

(i) $\mathrm{gr}^1 \mathcal{D}_4 = (0)$.

(ii) *For $m > 1$, let $f = f(x,y)$ run over all elements of $\mathrm{gr}^m \mathfrak{P}_4$ satisfying*

$$f(x,y) + f(y,x) = 0, \tag{2.2.1}$$

$$[y, f(x,y)] + [z, f(x,z)] = 0, \tag{2.2.2}$$

and for each such f, call $D_f = D_f^{(4)}$ the derivation of $\mathfrak{P}_4$ defined by

$$D_f(x) = 0, \quad D_f(y) = [y, f(x,y)]. \tag{2.2.3}$$

Then D_f is a y-normalized special derivation of $\mathfrak{P}_4$ of degree m which is S_4-invariant modulo inner derivations, and $f \to \{D_f\}$ gives a $\mathbf{Q}$-module isomorphism between the space of all $f \in \mathrm{gr}^m \mathfrak{P}_4$ satisfying (2.2.1) and (2.2.2) and the space $\mathrm{gr}^m \mathcal{D}_4$.

(iii) *From (2.2.1) and (2.2.2) follows the 3-cycle relation in $\mathfrak{P}_4$:*

$$f(x,y) + f(y,z) + f(z,x) = 0, \tag{2.2.4}$$

(iv) $[D_f, D_{f'}] = D_{f''}$, *with* $f'' = [f, f'] + D_f(f') - D_{f'}(f)$.

Proof: As these are essentially known ([1], [3], [4]), we shall only sketch the proof. Let $m > 1$ and $\{D\} \in \mathrm{gr}^m \mathcal{D}_4$, with D: y-normalized. Then $D(x) = 0$, $D(y) = [y, f]$ and $D(z) = [z, g]$, with some $f, g \in \mathrm{gr}^m \mathfrak{P}_4$. As $x + y + z = 0$, we have

$$[y, f] + [z, g] = 0. \tag{2.2.5}$$

Now S_4 acts on $\mathfrak{P}_4$ via its quotient $\simeq S_3$ as substitutions of x, y, z, and D is S_4-invariant modulo inner derivations. Hence

$$\sigma D \sigma^{-1} - D = \mathrm{Int}\, a(\sigma) \tag{2.2.6}$$

with some $a(\sigma) \in \mathrm{gr}^m \mathfrak{P}_4$ for each substitution σ of x, y, z.

First, take σ: $x \to x$, $y \leftrightarrow z$. Then the derivation (2.2.6) applied to x gives $[a(\sigma), x] = 0$; hence $a(\sigma) = 0$ (as $m > 1$). Therefore, (2.2.6) applied to y gives $[y, \sigma g - f] = 0$; hence $g = \sigma(f)$. This, together with (2.2.5), gives (2.2.2). Now take σ: $x \leftrightarrow y$, $z \to z$. Then (2.2.6) gives $a(\sigma) = f(x,y) = -f(y,x)$; hence

(2.2.1). Conversely, if f satisfies (2.2.1) (2.2.2), D_f is obviously y-normalized, special, and S_4-invariant modulo inner derivations. This settles (ii).

(iii) From (2.2.2), we obtain by changing variables:

$$[z, f(y,z)] + [x, f(y,x)] = 0. \tag{2.2.7}$$

By substracting (2.2.7) from (2.2.2) and using (2.2.1), we obtain

$$[x+y, f(x,y) + f(y,z) + f(z,x)] = 0;$$

hence (2.2.4).

(i) and (iv): Straightforward. ∎

2.3 Before proceeding to the case $n = 5$, we need:

Proposition 3: *If $1 \le i,j,k,l \le n$, $\{i,j\} \cap \{k,l\} = \phi$ and $z \in \mathfrak{P}_n$, then $[x_{ij}[x_{kl}, z]] = 0$ holds if and only if $z \in C_{ij} + C_{kl}$.*

Proof: First, we note that $[x_{ij}, x_{kl}] = 0$ and hence $[x_{ij}[x_{kl}, z]] = [x_{kl}[x_{ij}, z]]$. Now the "if" implication is obvious. To prove the other, assume $[x_{ij}[x_{kl}, z]] = 0$ and decompose z as $z = n_i + c_{ij}$ ($n_i \in N_i$, $c_{ij} \in C_{ij}$). By the assumption on $z, [x_{kl}, z] \in C_{ij}$. Also, clearly, $[x_{kl}, c_{ij}] \in C_{ij}$. Therefore, $[x_{kl}, n_i] \in C_{ij}$. But N_i being an ideal, $[x_{kl}, n_i] \in N_i$. Therefore, $[x_{kl}, n_i] = 0$ by (1.1.3). Therefore, $n_i \in C_{kl}$. Therefore, $z = n_i + c_{ij} \in C_{kl} + C_{ij}$. ∎

2.4 The case $n = 5$. In this section, we shall prove the "only if" implication of the following

Theorem 1: * *Let $f \in \mathrm{gr}^m \mathfrak{P}_4$ and $\{D_f^{(4)}\} \in \mathrm{gr}^m \mathcal{D}_4$ be as in Proposition 2. Then $\{D_f^{(4)}\}$ belongs to the image of $\psi_5 \colon \mathcal{D}_5 \to \mathcal{D}_4$ if and only if f satisfies the following 5-cycle relation in $\mathfrak{P}_5$:*

$$f(x_{12}, x_{23}) + f(x_{34}, x_{45}) + f(x_{51}, x_{12}) + f(x_{23}, x_{34}) + f(x_{45}, x_{51}) = 0. \tag{2.4.1}$$

Here, in general, for any Lie algebra $\mathcal{L}$ over $\mathbf{Q}$ and $a, b \in \mathcal{L}$, $f(a,b)$ denotes the image of f under the Lie homomorphism $\mathfrak{P}_4 \to \mathcal{L}$ defined by $x \to a$, $y \to b$.

The "if" implication in Theorem 1 will be proved in §4.

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* This theorem was obtained in 1988 and was used by Terada to check that some element of $\mathrm{gr}^7 \mathcal{D}_4$ is not extendable to $\mathrm{gr}^7 \mathcal{D}_5$.

Proof of the "only if" implication: Suppose that there exists $\{D\} \in \mathrm{gr}^m\mathcal{D}_5$ $(m > 1)$, with D: y-normalized, such that $\psi_5\{D\} = \{D_f^{(4)}\}$. As $y_2 = x_{12}$, $y_3 = x_{13} + x_{23}$ and $y_4 = -x_{45}$, we have

$$D(x_{12}) = D(x_{13} + x_{23}) = D(x_{45}) = 0.$$

[Claim] $$D(x_{23}) = [x_{23}, f(x_{12}, x_{23})].$$

Indeed, put $D(x_{23}) = [x_{23}, t_{23}]$, $t_{23} \in \mathrm{gr}^m\mathfrak{P}_5$. Since $[x_{23}, x_{45}] = 0$ and $D(x_{45}) = 0$, $[D(x_{23}), x_{45}] = 0$; hence $t_{23} \in C_{23} + C_{45}$, by Proposition 3. Thus, we may assume $t_{23} \in C_{45} = \langle x_{12}, x_{23}, x_{13}\rangle$ (the Lie subalgebra of $\mathfrak{P}_5$ generated by x_{12}, x_{23}, x_{13}). As $m > 1$ and $x_{12} + x_{23} + x_{13} = x_{45}$ is central in C_{45}, $t_{23} \in \langle x_{12}, x_{23}\rangle$. But since $\{D\}$ extends $\{D_f^{(4)}\}$, D must extend $D_f^{(4)}$, and hence the image of t_{23} on $\mathfrak{P}_4 \simeq \mathfrak{P}_5/N_5$ must be f. Therefore, $t_{23} = f(x_{12}, x_{23})$, whence the claim.

Now for each $\sigma \in S_5$,

$$\sigma D\sigma^{-1} - D = \mathrm{Int}\, a(\sigma)$$

with a unique $a(\sigma) \in \mathrm{gr}^m\mathfrak{P}_5$, and $\sigma \mapsto a(\sigma)$ is a 1-cocyle;

$$a(\sigma\tau) = a(\sigma) + \sigma a(\tau) \quad (\sigma, \tau \in S_5).$$

Put $\varepsilon = (15)(24)$, $\delta = (13524)$, $\rho = \varepsilon \circ \delta = (13)(45)$. Then, as ε maps as $y_2 \leftrightarrow -y_4$, $y_3 \leftrightarrow -y_3$, we have $a(\varepsilon) = 0$ by Proposition 1. As for ρ, ρ maps as $x_{12} \leftrightarrow x_{23}, x_{45} \to x_{45}$; hence $\rho D\rho^{-1} - D = \mathrm{Int}\, a(\rho)$ maps as:

$$x_{12} \to \rho[x_{23}, f(x_{12}, x_{23})] = [x_{12}, f(x_{23}, x_{12})], \quad x_{45} \to 0.$$

Therefore, $\mathrm{Int}\, a(\rho)$ coincides with $\mathrm{Int}\, f(x_{12}, x_{23})$ on y_2 and y_4 (and hence also on $y_3 = -y_2 - y_4$), and hence they coincide with each other by Proposition 1(i). Therefore,

$$a(\rho) = f(x_{12}, x_{23}).$$

Therefore, $a(\rho) = a(\varepsilon\delta) = a(\varepsilon) + \varepsilon \cdot a(\delta) = \varepsilon a(\delta)$; hence $a(\delta) = \varepsilon^{-1}a(\rho) = \varepsilon^{-1}f(x_{12}, x_{23}) = f(x_{45}, x_{34})$. Now since $a(\sigma)$ is a 1-cocycle and $\delta^5 = 1$, we have

$$(1 + \delta + \delta^2 + \delta^3 + \delta^4)f(x_{45}, x_{34}) = 0.$$

The desired formula (2.4.1) follows directly from this by using (2.2.1). ∎

3. More on 3- and 5-cycle relations

3.1 In order to be able to use the 5-cycle relation (2.4.1) fully, we need to understand the algebraic structure of the subset $\{x_{12}, x_{23}, \dots, x_{51}\}$ of $\mathfrak{P}_5$.

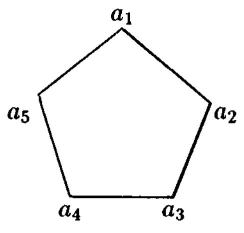
We shall prove:

PROPOSITION 4: *The Lie algebra $\mathfrak{P}_5$ is generated by $w_i = x_{i,i+1}$ $(i \in \mathbf{Z}/5 \approx \{1, 2, \dots, 5\})$, and the defining relations among the w_i are:*

(3.1.1) $$[w_i, w_j] = 0 \quad \text{if} \quad i - j \not\equiv \pm 1 \pmod 5,$$

(3.1.2) $$\sum_i [w_i, w_{i+1}] = 0.$$

For any Lie algebra $\mathcal{L}$ over $\mathbf{Q}$ and $a_i \in \mathcal{L}$ $(i \in \mathbf{Z}/5)$, we say that the a_i's form an *admissible pentagon*



if (3.1.1) and (3.1.2) are satisfied for the a_i in place of the w_i. Note that if $\{a_i\}$ forms an admissible pentagon then so does $\{a_{-i}\}$.

COROLLARY 1: *There exists a Lie homomorphism $\varphi \colon \mathfrak{P}_5 \to \mathcal{L}$ such that $\varphi(w_i) = a_i$ $(i \in \mathbf{Z}/5)$ if and only if $\{a_i\}_{i \in \mathbf{Z}/5}$ forms an admissible pentagon.*

COROLLARY 2: *If $f(x, y) \in \mathfrak{P}_4$ satisfies the 5-cycle relation (2.4.1), and $\{a_i\}_{i \in \mathbf{Z}/5}$ forms an admissible pentagon, then*

$$\sum_{i \in \mathbf{Z}/5} f(a_i, a_{i+1}) = 0.$$

Proof of Proposition 4:
(i): *That $\mathfrak{P}_5$ is generated by the w_i.* This is clear by the formula (a special case of (1.1.1))

(3.1.3) $$x_{i,i+2} = x_{i+3,i+4} - x_{i,i+1} - x_{i+1,i+2}$$

$(i \in \mathbf{Z}/5)$.

(ii) *That the w_i's satisfy* (3.1.1) and (3.1.2): (3.1.1) is obvious, and (3.1.2) follows directly from

$$[x_{45} - x_{12} - x_{23},\ x_{51} - x_{23} - x_{34}] = [x_{13}, x_{24}] = 0.$$

(iii) *That* (3.1.1) *and* (3.1.2) *are the fundamental relations*: Since $\dim \mathrm{gr}^1\mathfrak{P}_5 = 5$, we only need to show that the *quadratic* relations $[x_{ij}, x_{kl}] = 0$ ($\{i,j\} \cap \{k,l\} = \phi$) follow from (3.1.1) and (3.1.2). When either $i - j \equiv \pm 1$ or $k - l \equiv \pm 1 \pmod 5$, this relation follows directly from (3.1.1) (using (3.1.3) as definition of $x_{ij} = x_{ji}$ when $i - j \equiv \pm 2$). When $i - j \equiv \pm 2$ *and* $k - l \equiv \pm 2$, we may assume $k = i + 1,\ j = i + 2,\ l = i + 3$, so that

$$x_{ij} = x_{i+3,i+4} - x_{i,i+1} - x_{i+1,i+2},$$
$$x_{kl} = x_{i,i+4} - x_{i+1,i+2} - x_{i+2,i+3}.$$

In this case, $[x_{ij}, x_{kl}] = 0$ follows from (3.1.1) and (3.1.2). ∎

3.2 Let $f = f(x,y) \in \mathrm{gr}^m\mathfrak{P}_4$ ($m > 1$), $\mathcal{L}$ be any Lie algebra over $\mathbf{Q}$, and $a, b, c \in \mathcal{L}$.

PROPOSITION 5:

(i) *If c commutes with a and b, then $f(a,b) = f(a+c,b) = f(a,b+c)$;*

(ii) *If f satisfies (2.2.2) (resp. (2.2.4)) and $a+b+c$ commutes with a,b,c, then*

$$[b, f(a,b)] + [c, f(a,c)] = 0$$
$$(\text{resp. } f(a,b) + f(b,c) + f(c,a) = 0).$$

Proof:

(i) Clear, as $m > 1$.

(ii) If $a+b+c = 0$, then this is obvious. The point is that we only need $a+b+c$ to be commutative with a,b,c. To see this, let $\mathfrak{P}^*$ be the Lie algebra over $\mathbf{Q}$ generated by ξ, η, ζ with the defining relation: $\xi + \eta + \zeta$ *commutes with* ξ, η, ζ. Then $\mathfrak{P}^*/\mathbf{Q}\cdot(\xi + \eta + \zeta) \tilde{\to} \mathfrak{P}_4$, and $f(\xi,\eta) + f(\eta,\zeta) + f(\zeta,\xi)$ and $[\eta, f(\xi,\eta)] + [\zeta, f(\xi,\zeta)]$ have 0 as their images on $\mathfrak{P}_4$. But since $\deg f > 1$, they themselves must be 0. The rest is obvious. ∎

We shall say that a, b, c form an **admissible triangle** if $a + b + c$ commutes with a, b, c.

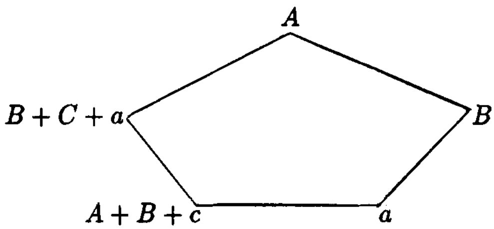
3.3 PROPOSITION 6: *Let A, B, C, a, b, c be six elements of a Lie algebra $\mathcal{L}$ over $\mathbf{Q}$ satisfying*

(i) $[A, a] = [B, b] = [C, c] = 0$,

(ii) *each of* $\{A, B, c\}, \{A, b, C\}, \{a, B, C\}$ *is an admissible triangle.*

Then

(3.3.1)



is an admissible pentagon, and so is any S_3-transform of (3.3.1) obtained by interchanging the ordered pairs $(A, a), (B, b), (C, c)$.

Proof: Since the assumptions on A, B, C, a, b, c are S_3-symmetric, it suffices to show that (3.3.1) is admissible. First it is clear that the elements corresponding to non-adjacent vertices commute with each other. Secondly,

$$\begin{aligned}[B, a] + [a, A + B + c] + [A + B + c, B + C + a]\\ + [B + C + a, A] + [A, B]\\ = [a, c] + [A + c, C + a] + [C + a, A] = 0. \quad \blacksquare\end{aligned}$$

PROPOSITION 7: *Let $A, B, C, a, b, c \in \mathcal{L}$ satisfy, in addition to the conditions (i) and (ii) of Proposition 6,*

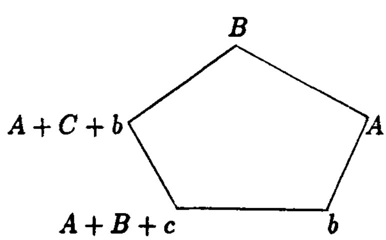
(iii) $\{a, b, c\}$ *is an admissible triangle.*

Then, for any $f \in \mathrm{gr}^m \mathfrak{P}_4$ $(m > 1)$ satisfying the 2,3,5-cycle relations (2.2.1), (2.2.4), and (2.4.1),

$$(3.3.2)\quad \begin{aligned} f(A, B) + f(B, C) + f(C, A)\\ = f(A + b, B + a) + f(B + c, C + b) + f(C + a, A + c).\end{aligned}$$

Proof: Use the admissible pentagon

(3.3.3)



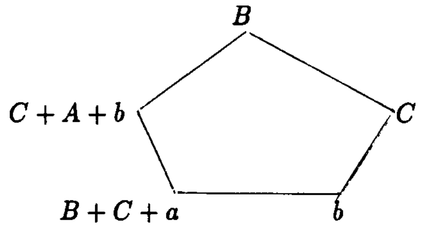
(obtained from (3.3.1) by the transposition $\{A,a\}\leftrightarrow\{B,b\}$) and Proposition 5 (i) for $A\leftrightarrow B+c$, $C+b$, and (2.2.1), to derive:

$$(3.3.4)\quad f(A,B)+f(C+b,B+c)=f(A+C+b,B)+f(b,A+B+c)+f(A,b).$$

By Proposition 5 (ii) applied to the admissible triangle $\{A,b,C\}$, and by (2.2.1), we obtain

$$f(C,A)+f(A,b)=f(C,b). \tag{3.3.5}$$

Also,



is admissible; hence

$$(3.3.6)\quad \begin{aligned} &f(B,C)+f(C,b)+f(b,B+C+a)\\ &\quad+f(B+C+a,C+A+b)+f(C+A+b,B)=0.\end{aligned}$$

By adding both sides of (3.3.4)$\sim$(3.3.6) we obtain

$$\begin{aligned} &f(A,B)+f(B,C)+f(C,A)\\ &\quad=f(B+c,C+b)+f(b,A+B+c)+f(B+C+a,b)\\ &\qquad+f(A+C+b,B+C+a).\end{aligned}$$

But the sum of the second and the third terms on the RHS

$$=f(b,A+c)+f(C+a,b)=f(C+a,A+c),$$

because $\{b,A+c,C+a\}$ is an admissible triangle (by (i)$\sim$(iii)). Finally, as C commutes with $A+b$ and $B+a$, C can be dropped off from the last term on the RHS. ∎

3.4 The above Propositions 5, 6, 7 will be applied later to the following case.

PROPOSITION 8: *Let M be a non-empty subset of $\{1,2,\ldots,n\}$, and i,j,k be distinct indices from $\{1,\ldots,n\}$ not belonging to M. Put*

$$x_{iM} = \sum_{m\in M} x_{im},$$

and define x_{jM}, x_{kM} similarly. Then the system

$$\begin{cases} A = x_{iM}, & B = x_{jM}, \quad C = x_{kM}, \\ a = x_{jk}, & b = x_{ki}, \quad c = x_{ij} \end{cases} \tag{3.4.1}$$

in $\mathfrak{P}_n$ satisfies the conditions (i)(ii) of Proposition 6 and (iii) of Proposition 7. In particular, if $f \in \mathrm{gr}^m\mathfrak{P}_4$ $(m > 1)$ satisfies the 2,3,5 cycle relations, then f satisfies (3.3.2) and also

$$\begin{aligned} f(A,B) + f(B,a) + f(a,A+B+c) + f(A+B+c,B+C+a)& \\ + f(B+C+a,A) = 0.& \end{aligned} \tag{3.4.2}$$

Proof: We only note that

$$B + c = -x_{jM'}, \quad A + B = \sum_{m\in M}(x_{im} + x_{jm}),$$

M' being the complement of $M \cup \{i\}$ in $\{1,\cdots,n\}$. These make it clear that $B + c$ commutes with $A = x_{iM}$ and that $c = x_{ij}$ commutes with $A + B$, and hence that $\{A,B,c\}$ forms an admissible triangle. The rest is obvious. ∎

4. Extendability

4.1 Now let $m > 1$ and $f = f(x,y) \in \mathrm{gr}^m\mathfrak{P}_4$ satisfy (2.2.1), (2.2.2), and (2.2.4)(in $\mathfrak{P}_4$) and (2.4.1) (in $\mathfrak{P}_5$):

$$f(x,y) + f(y,x) = 0, \tag{2.2.1}$$

$$[y, f(x,y)] + [z, f(x,z)] = 0, \tag{2.2.2}$$

$$f(x,y) + f(y,z) + f(z,x) = 0, \tag{2.2.4}$$

$$\sum_{i\in\mathbf{Z}/5} f(x_{i,i+1}, x_{i+1,i+2}) = 0. \tag{2.4.1}$$

Let $D_f^{(4)}$ be the derivation of $\mathfrak{P}_4$ defined in Proposition 2. Our goal is to show that for each $n \geq 5$, $\{D_f^{(4)}\}$ extends to an element $\{D_f^{(n)}\}$ of $\mathcal{D}_n$. (Recall that

(2.4.1) is a necessary condition for the extendability of $\{D_f^{(4)}\}$ to $\mathcal{D}_5$ (§2.4).) We can write down the formula for $D_f^{(n)}$ explicitly (see Theorem 2 and Proposition 9 below), but to prove that this formula really gives a well-defined derivation, etc., it is technically easier to construct first the corresponding 1-cocycle $a_f(\sigma)$ with respect to the S_n-action on $\mathrm{gr}^m\mathfrak{P}_n$, connected to $D_f^{(n)}$ by the formula

$$\sigma D_f^{(n)}\sigma^{-1} - D_f^{(n)} = \mathrm{Int}\, a_f(\sigma) \quad (\sigma \in S_n).$$

We begin with this construction.

For each i $(1 \le i \le n-1)$, call σ_i the transposition $\sigma_i = (i, i+1) \in S_n$.

KEY LEMMA: *There exists a unique 1-cocycle $S_n \to \mathrm{gr}^m\mathfrak{P}_n$ $(\sigma \mapsto a_f(\sigma))$ such that*

$$a_f(\sigma_1) = a_f(\sigma_{n-1}) = 0,$$
$$a_f(\sigma_i) = f(y_i, y_{i+1} - x_{i,i+1}) \quad (2 \le i \le n-2).$$

Proof: Since the σ_i's generate S_n, such a 1-cocycle is unique if exists at all. The existence relies heavily on the conditions (2.2.1), (2.2.4), and (2.4.1) satisfied by f, as we shall see.

As S_n is generated by the σ_i's and the fundamental relations are

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| > 1),$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (1 \le i \le n-2),$$
$$\sigma_i^2 = 1 \quad (1 \le i \le n-1),$$

it suffices to prove the following (i)~(iii):

(i) $a_f(\sigma_i)$ is σ_j-invariant if $|i-j| > 1$,

(ii) $a_f(\sigma_i) + \sigma_i a_f(\sigma_{i+1}) + \sigma_i\sigma_{i+1}a_f(\sigma_i)$
$= a_f(\sigma_{i+1}) + \sigma_{i+1}a_f(\sigma_i) + \sigma_{i+1}\sigma_i a_f(\sigma_{i+1}) \quad (1 \le i \le n-2)$,

(iii) $(1+\sigma_i)a_f(\sigma_i) = 0 \quad (1 \le i \le n-1)$.

Proof of (i): If $j < i-1$ or $j > i+1$, then σ_j leaves y_i, y_{i+1} and $x_{i,i+1}$ invariant; hence σ_j also leaves $a_f(\sigma_i) = f(y_i, y_{i+1} - x_{i,i+1})$ invariant.

Proof of (ii): If we write $j = i+1$, $k = i+2$ and $M = \{1, 2, \ldots, i-1\}$, then (with the notation of Proposition 8),

$$\begin{aligned}\text{LHS of (ii)} =& f(y_i, y_{i+1} - x_{i,i+1}) + f(y_i + x_{i,i+1}, y_{i+2} - x_{i,i+2}) \\ &+ f(y_{i+1} - x_{i,i+1}, y_{i+2} - x_{i,i+2} - x_{i+1,i+2}) \\ =& f(x_{iM}, y_{jM}) + f(x_{iM} + x_{ij}, x_{kM} + x_{jk}) + f(x_{jM}, x_{kM}) \\ =& f(A,B) + f(A+c, C+a) + f(B,C),\end{aligned}$$

and

$$\begin{aligned}\text{RHS of (ii)} =& f(y_{i+1}, y_{i+2} - x_{i+1,i+2}) + f(y_i, y_{i+2} - x_{i,i+2} - x_{i+1,i+2}) \\ &+ f(y_i + x_{i,i+2}, y_{i+1} - x_{i,i+1} + x_{i+1,i+2}) \\ =& f(x_{jM}+x_{ij}, x_{kM}+x_{ki}) + f(x_{iM}, x_{kM}) + f(x_{iM}+x_{ki}, x_{jM}+x_{jk}) \\ =& f(B+c, C+b) + f(A,C) + f(A+b, B+a).\end{aligned}$$

Therefore, they are equal by Propositions 7, 8.

Proof of (iii): $(1+\sigma_i)a(\sigma_i) = f(y_i, y_{i+1} - x_{i,i+1}) + f(y_{i+1} - x_{i,i+1}, y_i) = 0.$ ∎

4.2 Consider the subgroup $S_2 \times S_{n-2} \subset S_n$ generated by σ_i $(1 \le i \le n-1, i \ne 2)$. Let f, $a_f(\sigma)$ be as in §4.1. Then

$$a_f(\sigma) \in C_{12} \quad \text{for} \quad \sigma \in S_2 \times S_{n-2}.$$

Indeed, $a_f(\sigma_i) \in C_{12}$ for $i \ne 2$ (as $y_i, x_{i,i+1} \in C_{12}$ for $i \ge 3$), and C_{12} is $(S_2 \times S_{n-2})$-stable. Therefore, if $1 \le i,j \le n$ $(i \ne j)$, and $\sigma \in S_n$ is such that $\sigma(1) = i$, $\sigma(2) = j$, then $a_f(\sigma) \bmod C_{ij}$ is independent of the choice of σ. Call this class f_{ij}. Our goal is to prove:

THEOREM 2: *The notation being as above, $D_f^{(n)}$: $x_{ij} \to [x_{ij}, f_{ij}]$ $(1 \le i,j \le n, i \ne j)$ defines a y-normalized special derivation of $\mathfrak{P}_n$ which extends the derivation $D_f^{(4)}$ of $\mathfrak{P}_4$ and which satisfies*

$$\sigma D_f^{(n)} \sigma^{-1} - D_f^{(n)} = \operatorname{Int} a_f(\sigma) \quad (\sigma \in S_n).$$

First, we shall prove:

PROPOSITION 9: *If $i < j$, then*

$$(4.2.1) \qquad f_{ij} \equiv f(y_i, x_{ij}) + \sum_{l=i+1}^{j-1} f(y_l, \sum_{k=1}^{l-1} x_{kj}) \pmod{C_{ij}} \qquad (y_1 = 0).$$

Proof:

(i) *The case* $j = i + 1$. We shall prove

$$f_{i,i+1} \equiv f(y_i, x_{i,i+1}) \pmod{C_{i,i+1}} \tag{4.2.2}$$

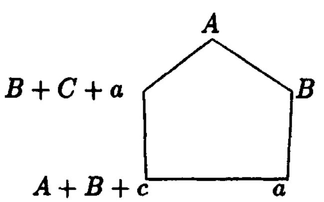
by induction on i. If $i = 1$, both sides are 0. Assume (4.2.2) for some $i \leq n-2$. Then, as $\sigma_i\sigma_{i+1}$ maps $i, i+1$ to $i+1, i+2$ respectively,

$$\begin{aligned} f_{i+1,i+2} &\equiv a(\sigma_i\sigma_{i+1}) + (\sigma_i\sigma_{i+1})f_{i,i+1} \\ &\equiv a(\sigma_i) + \sigma_i a(\sigma_{i+1}) + (\sigma_i\sigma_{i+1})f_{i,i+1} \\ &\equiv f(y_i, y_{i+1} - x_{i,i+1}) + f(y_i + x_{i,i+1}, y_{i+2} - x_{i,i+2}) \\ &\quad + f(y_{i+1} - x_{i,i+1}, x_{i+1,i+2}) \end{aligned}$$

$(\mathrm{mod}\ C_{i+1,i+2})$. Therefore,

$$\begin{aligned} f_{i+1,i+2} &\equiv f(A,B) + f(A+c, C+a) + f(B,a) \\ &= f(A,B) + f(A+B+c, B+C+a) + f(B,a), \end{aligned}$$

where $A = x_{iM}$, $B = x_{i+1,M}$, $C = x_{i+2,M}$, $a = x_{i+1,i+2}$, $b = x_{i,i+2}$, $c = x_{i,i+1}$, with $M = \{1, \ldots, i-1\}$. But



is admissible (Propositions 6, 8); hence

$$\begin{aligned} f_{i+1,i+2} &\equiv f(A+B+c, a) + f(A, B+C+a) \\ &= f(B+c, a) + f(A, B+C) \\ &\equiv f(B+c, a) \pmod{C_{i+1,i+2}} \\ &= f(y_{i+1}, x_{i+1,i+2}) \pmod{C_{i+1,i+2}}, \end{aligned}$$

because A and $B+C$ commutes with $a = x_{i+1,i+2}$. This settles the case (i).

(ii) *The general case* $j \geq i+1$. Induction on j. Apply σ_j on (4.2.1) to get

$$f_{i,j+1} - a_f(\sigma_j) \equiv f(y_i, x_{i,j+1}) + \sum_{l=i+1}^{j-1} f(y_l, \sum_{k=1}^{l-1} x_{k,j+1}) \ (\mathrm{mod} C_{i,j+1}),$$

which gives

$$f_{i,j+1} \equiv f(y_i, x_{i,j+1}) + \sum_{l=i+1}^{j} f(y_l, \sum_{k=1}^{l-1} x_{k,j+1}). \qquad \blacksquare$$

4.3 PROOF OF THEOREM 2

(I) That $D_f^{(n)}$: $x_{ij} \to [x_{ij}, f_{ij}]$ defines a derivation of $\mathfrak{P}_n$. To prove this, it suffices to check:

(i) $$f_{ij} \equiv f_{ji} \; (\mathrm{mod} C_{ij}),$$

(ii) $$\sum_{i=1}^{n} [x_{ij}, f_{ij}] = 0 \quad (1 \leq j \leq n),$$

(iii) $$f_{ij} - f_{kl} \in C_{ij} + C_{kl} \quad \text{if} \quad \{i,j\} \cap \{k,l\} = \phi$$

(cf. Proposition 3).

Proofs of (i), (ii), and (iii):

(i) $a_f(\sigma\sigma_1) = a_f(\sigma) + \sigma a_f(\sigma_1) = a_f(\sigma)$ for any $\sigma \in S_n$.

(ii) For each $j \geq 2$,

$$S_j := \sum_{i=1}^{j-1} [x_{ij}, f_{ij}] = \sum_{i=1}^{j-1} [x_{ij}, f(y_i, x_{ij})] + \sum_{i=1}^{j-1} \sum_{l=i+1}^{j-1} [x_{ij}, f(y_l, \sum_{k=1}^{l-1} x_{kj})].$$

By changing the order of summation in the second term on the RHS, we obtain

$$S_j = \sum_{l=2}^{j-1} \{[x_{lj}, f(y_l, x_{lj})] + [\sum_{k=1}^{l-1} x_{kj}, f(y_l, \sum_{k=1}^{l-1} x_{kj})]\}.$$

But since x_{lj}, y_l and $\sum_{k=1}^{l-1} x_{kj}$ form an admissible triangle, each summand in the above expression for S_j must be 0 by Proposition 5 (ii). Therefore, $S_j = 0$.

In particular, for $j = n$,

$$\sum_{\nu \neq n} [x_{\nu n}, f_{\nu n}] = 0.$$

Now let j be any index $(1 \leq j \leq n)$ and $\sigma \in S_n$ be such that $\sigma(n) = j$. Then $\sigma f_{\nu n} \equiv f_{\mu j} - a_f(\sigma) \; (\mathrm{mod} C_{\mu j})$, where $\mu = \sigma(\nu)$, and $\sum_{\mu \neq j} x_{\mu j} = 0$; hence

$$\sum_{\mu \neq j} [x_{\mu j}, f_{\mu j}] = 0.$$

This settles (ii).

(iii) It suffices to prove this for *one* choice of a quadruple $\{i,j,k,l\}$. This is because S_n acts transitively on such quadruples and

$$f_{\sigma i,\sigma j} \equiv \sigma f_{ij} + a_f(\sigma) \mod C_{\sigma i,\sigma j},$$
$$f_{\sigma k,\sigma l} \equiv \sigma f_{kl} + a_f(\sigma) \mod C_{\sigma k,\sigma l}.$$

Choose $\{i,j\} = \{1,2\}$, $\{k,l\} = \{n-1,n\}$. Then $f_{12} \equiv 0$ (mod C_{12}) (obvious), and $f_{n-1,n} \equiv 0$ (mod $C_{n-1,n}$) by Proposition 9 (because $y_{n-1} = -x_{n-1,n}$). Therefore, $f_{1,2} - f_{n-1,n} \in C_{12} + C_{n-1,n}$.

Therefore, $D_f^{(n)}$ defines a derivation of $\mathfrak{P}_n$, which is obviously special. Write $D = D_f^{(n)}$.

(II) Since we have shown above that $S_j = 0$, we have $D(y_j) = 0$ $(2 \le j \le n-1)$. Therefore, D is y-normalized.

(III) For each k,l $(1 \le k,l \le n)$, $k \ne l$, choose $\tau_{kl} \in S_n$ which map 1,2 to k,l respectively. Then $D(x_{kl}) = [x_{kl}, a_f(\tau_{kl})]$. For each i $(1 \le i \le n-1)$, consider the derivation $\sigma_i D \sigma_i^{-1} - D$ of $\mathfrak{P}_n$. Then this maps x_{kl} to $[x_{kl}, \sigma_i(a_f(\sigma_i^{-1}\tau_{kl})) - a_f(\tau_{kl})] = [x_{kl}, -a_f(\sigma_i)] = [a_f(\sigma_i), x_{kl}]$, for any k,l. Therefore, $\sigma_i D \sigma_i^{-1} - D = \operatorname{Int} a_f(\sigma_i)$.

(IV) Finally, since $D(x_{23}) = [x_{23}, t_{23}] = [x_{23}, f(y_2, x_{23})] = [x_{23}, f(x_{12}, x_{23})]$, and $D(x_{12}) = 0$, D extends $D_f^{(4)}$. ∎

4.4 From Theorem 2, the "if" implication of Theorem 1, as well as the Main Theorem (§1.2), follow immediately.

Remark: Drinfeld shows, in a slightly different language (plane braids on 4 strings instead of sphere braids on 5 strings) that (2.2.2) follows from (2.2.1), (2.2.4), and (2.4.1) (see [1] §5 (Proposition 5.7)).

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